Multi-Source Noisy Network Coding

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***Abstract*—Noisy network coding unifies network coding by Ahlswede, Cai, Li, and Yeung for noiseless networks and compress–forward by Cover and El Gamal for noisy relay channels. In particular, it achieves the best known capacity inner bounds for multi-source multicast networks including deterministic networks by Avestimehr, Diggavi, and Tse and erasure networks by Dana, Gowaikar, Palanki, Hassibi, and Effros. This paper extends noisy network coding for multicast networks to networks with general message demand by combining the underlying noisy network coding scheme with decoding techniques for interference channels. At one extreme, noisy network coding is combined with simultaneous decoding, while at the other extreme interference is treated as noise. The potential of noisy network coding as a canonical building block for wireless networks is demonstrated via three examples of Gaussian networks that have drawn recent attentions.**

I. Introduction and Main Results

An N-node discrete memoryless network (DMN)

$$(\mathcal{X}_1\times\cdots\times\mathcal{X}_N, p(y_1,\ldots,y_N|x_1,\ldots,x_N), \mathcal{Y}_1\times\cdots\times\mathcal{Y}_N) \quad (1)$$

consists of N sender–receiver alphabet pairs $(\mathcal{X}_k,\mathcal{Y}_k)$, $k\in[1:N]:=\{1,\ldots,N\}$, and a collection of conditional pmfs $p(y_1,\ldots,y_N|x_1,\ldots,x_N)$. Each node $k\in[1:N]$ wishes to send a message M_k to a set $\mathcal{D}_k$ of destination nodes. Formally, a $(2^{nR_1},\ldots,2^{nR_N},n)$ code for a DMN consists of N message sets $[1:2^{nR_1}],\ldots,[1:2^{nR_N}]$, a set of encoders with encoder $k\in[1:N]$ that assigns an input symbol x_{ki} to each pair (m_k,y_k^{i-1}) for $i\in[1:n]$, and a set of decoders with decoder $d\in\cup_{k=1}^N\mathcal{D}_k$ that assigns message estimates $\{\hat{m}_{kd}:k\in\mathcal{S}_d\}$ to each (y_d^n,m_d), where $\mathcal{S}_d:=\{k:d\in\mathcal{D}_k\}$ is the set of nodes that send messages to destination d.

We assume that messages M_k, $k\in[1:N]$, are each uniformly distributed over $[1:2^{nR_k}]$, and are independent of each other. The average probability of error is defined by

$$P_e^{(n)}=\mathsf{P}\{\hat{M}_{kd}\neq M_k \text{ for some } d\in\mathcal{D}_k, k\in[1:N]\}.$$

A rate tuple $(R_1,\ldots,R_N)$ is said to be achievable if there exists a sequence of $(2^{nR_1},\ldots,2^{nR_N},n)$ codes with $P_e^{(n)}\to 0$ as $n\to\infty$. The capacity region of the DMN is the closure of the set of achievable rate tuples.

A computable characterization of the capacity region is not known in general. The following cutset outer bound [1] provides a necessary condition for reliable communication:

If the rate tuple $(R_1,\ldots,R_N)$ is achievable, then there exists some joint pmf $p(x_1,\ldots,x_N)$ such that

$$\sum_{k\in\mathcal{S}:\mathcal{D}_k\cap\mathcal{S}^c\neq\emptyset} R_k\le I(X(\mathcal{S});Y(\mathcal{S}^c)|X(\mathcal{S}^c)) \quad (2)$$

for all $\mathcal{S}\subseteq[1:N]$ such that $\mathcal{D}_k\cap\mathcal{S}^c\neq\emptyset$.

There are several special cases for which the capacity region is known. In the seminal paper on network coding [2], Ahlswede, Cai, Li, and Yeung established the capacity for the single-source multicast case ($R_k=0$ for $k\neq 1$) when the network is *noiseless*, that is, when it can be represented by a directed graph $(\mathcal{N},\mathcal{E})$ with capacity limited links. They showed that capacity coincides with the cutset bound, generalizing the max-flow min-cut theorem [3], [4] to multiple destinations. Each relay in network coding sends a function of its incoming signals over each outgoing link instead of simply forwarding incoming signals. Dana, Gowaikar, Palanki, Hassibi, and Effros [5] showed that network coding is also optimal for noiseless multi-source multicast networks ($\mathcal{D}_1=\cdots=\mathcal{D}_N=\mathcal{D}$). These results have been further extended to multicast scenarios over erasure networks [5] and deterministic networks [6].

Recently, we have proposed noisy network coding for multicast, which combines both network coding and compress–forward by Cover and El Gamal [7] for the relay channel.

Theorem 1 ([8]): Suppose $\mathcal{D}_k=\mathcal{D}$ for $k\in[1:N]$. A rate tuple $(R_1,\ldots,R_N)$ is achievable if there exists some joint pmf $p(q)\prod_{k=1}^N p(x_k|q)p(\hat{y}_k|y_k,x_k,q)$ such that

$$\begin{aligned}R(\mathcal{S})<\min_{d\in\mathcal{S}^c\cap\mathcal{D}}&(I(X(\mathcal{S});\hat{Y}(\mathcal{S}^c),Y_d|X(\mathcal{S}^c),Q)\\&-I(Y(\mathcal{S});\hat{Y}(\mathcal{S})|X^N,\hat{Y}(\mathcal{S}^c),Y_d,Q))\end{aligned} \quad (3)$$

for all *cutsets* $\mathcal{S}\subseteq[1:N]$ with $\mathcal{S}^c\cap\mathcal{D}\neq\emptyset$, where $R(\mathcal{S})=\sum_{k\in\mathcal{S}}R_k$.

It can be readily checked that Theorem 1 recovers as special cases aforementioned results including coding for wireless relay networks and deterministic networks by Avestimehr, Diggavi, and Tse [6], and coding for wireless erasure networks by Dana, Gowaikar, Palanki, Hassibi, and Effros [5]. Compared to the cutset bound, the inner bound (3) in Theorem 1 has the first term with Y replaced by the "compressed" version $\hat{Y}$, the additional negative term that quantifies the rate requirement to convey the compressed version, and the

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maximum over independent X^N.

The key idea is to use block Markov message repetition coding and simultaneous decoding. Instead of sending multiple independent messages over several blocks and decoding them sequentially as in previous relaying schemes, the same message is sent multiple times using independent codebooks and the decoder performs joint typicality decoding on the received signals from all the blocks without explicitly decoding the compression indices.

In the general message demand (nonmulticast) settings, we note that Theorem 1 continues to hold with *multicast completion* of destination nodes, i.e., requiring every message to be decoded by all destination nodes $\mathcal{D} = \cup_{k=1}^N \mathcal{D}_k$. Thus, as a direct extension to Theorem 1, we can obtain an inner bound on the capacity region for the DMN in the same form as (3) with $\mathcal{D} = \cup_{k=1}^N \mathcal{D}_k$.

In this paper, we further extend Theorem 1 to general message demand settings. Noisy network coding can be viewed as transforming a relay network $p(y_1, \ldots, y_N|x_1, \ldots, x_N)$ into an interference network $p(\tilde{y}_1, \ldots, \tilde{y}_N|x_1, \ldots, x_N)$, where the effective channel output at decoder k is given by $\tilde{Y}_k = (X_k, Y_k, \hat{Y}_1, \ldots, \hat{Y}_N)$ and the compressed channel observations $(\hat{Y}_1, \ldots, \hat{Y}_N)$ are conveyed to decoders with some rate penalty. Thus, we can further incorporate known coding techniques for interference channels [9] to the noisy network coding. In a sense, the multicast completion inner bound corresponds to the capacity inner bound on the interference channel that is characterized by the intersection of the capacity regions of the multiple access channels, in which each decoder decodes all messages. By relaxing the decoding procedure for each destination node to correctly decode the intended messages only, we can obtain the following:

Theorem 2: A rate tuple $(R_1, \ldots, R_N)$ is achievable for the DMN if there exists some joint pmf $p(q)\prod_{k=1}^N p(x_k|q)p(\hat{y}_k|y_k, x_k, q)$ such that

$$\begin{aligned} R(\mathcal{S}) < \min_{d \in \mathcal{S}^c \cap \mathcal{D}(\mathcal{S})} & I(X(\mathcal{S}); \hat{Y}(\mathcal{S}^c), Y_d | X(\mathcal{S}^c), Q) \\ & - I(Y(\mathcal{S}); \hat{Y}(\mathcal{S}) | X^N, \hat{Y}(\mathcal{S}^c), Y_d, Q) \end{aligned} \quad (4)$$

for all cutsets $\mathcal{S} \subseteq [1:N]$ with $\mathcal{S}^c \cap \mathcal{D}(\mathcal{S}) \neq \emptyset$, where $\mathcal{D}(\mathcal{S}) := \cup_{k \in \mathcal{S}} \mathcal{D}_k$ and $R(\mathcal{S}) = \sum_{k \in \mathcal{S}} R_k$.

For the detailed description of the coding scheme and its analysis, refer to [8].

Since $\mathcal{D}(\mathcal{S}) \subset \mathcal{D}$, Theorem 2 gives a tighter inner bound on the capacity region. It is easy to find an example for which this improvement is strict (for example, consider two orthogonal noiseless links).

At the other extreme of dealing with interference from unintended messages, the decoders can ignore interference as noise rather than decoding it. This coding scheme results in the following:

Theorem 3: A rate tuple $(R_1, \ldots, R_N)$ is achievable for the DMN if there exists some joint pmf $p(q)\prod_{k=1}^N p(x_k, u_k|q)p(\hat{y}_k|y_k, u_k, q)$ with

$$\begin{aligned} R(\mathcal{T}) < & I(X(\mathcal{T}), U(\mathcal{S}); \hat{Y}(\mathcal{S}^c), Y_d | X(\mathcal{T}^c), U(\mathcal{S}^c), Q) \\ & - I(Y(\mathcal{S}); \hat{Y}(\mathcal{S}) | X(\mathcal{S}_d), U^N, \hat{Y}(\mathcal{S}^c), Y_d, Q) \end{aligned} \quad (5)$$

for all $d \in \cup_{k=1}^N \mathcal{D}_k$, $\mathcal{S} \subseteq [1:N]$, and $\mathcal{T} \subseteq \mathcal{S}_d$ such that $d \in \mathcal{S}^c$, $\mathcal{S} \cap \mathcal{S}_d \subseteq \mathcal{T}$ where $\mathcal{T}^c := \mathcal{S}_d \backslash \mathcal{T}$.

Unlike the previous noisy network coding schemes, each node uses superposition coding to send the compression index and the message. Again, for the detailed description of the coding scheme and its analysis, refer to [8].

II. Gaussian Network

Motivated by wireless networks, we consider the additive white Gaussian noise (AWGN) network in which the channel outputs are given by

$$Y^N = GX^N + Z^N, \quad (6)$$

where $G \in \mathbb{R}^{N \times N}$ is the channel gain matrix and Z^N is a vector of independent white Gaussian noise processes with zero mean and unit variance. We assume average power constraint P on each sender, i.e.,

$$\sum_{i=1}^n \mathsf{E}\left(x_{ki}^2(m_k, Y_k^{i-1})\right) \leq nP$$

for all $k \in [1:N]$ and $m_k \in [1:2^{nR_k}]$.

In the following subsections, we evaluate the performance of noisy network coding for three Gaussian networks. We first consider the Gaussian multiple-source multicast network, and we establish an inner bound that improves upon previous capacity approximation results by Avestimehr, Diggavi, and Tse [10] and Perron [11] with a tighter gap to the cutset bound. We then show that noisy network coding can outperform other specialized schemes for two-way relay channels [12], [13] and interference relay channels [14], [15].

A. AWGN-multiple multicast network

Theorem 4: Let $\mathcal{D} = \mathcal{D}_1 = \cdots = \mathcal{D}_N$. For any rate tuple $(R_1, \ldots, R_N)$ in the cutset outer bound, there exists $(R'_1, \ldots, R'_N)$ in the inner bound in Theorem 1 for the AWGN network (6) such that

$$\sum_{k \in \mathcal{S}} (R_k - R'_k) \leq \frac{|\mathcal{S}|}{2} + \frac{\min\{|\mathcal{S}|, |\mathcal{S}^c|\}}{2} \log(2|\mathcal{S}|)$$

for all $\mathcal{S} \subseteq [1:N]$ with $\mathcal{S}^c \cap \mathcal{D} \neq \emptyset$.

This theorem implies that the gap between the cutset bound and our inner bound is less than or equal to $(N/4)\log(2N)$ for $N > 3$, regardless of the values of the channel gain matrix G and power constraint P.

The cutset outer bound for the AWGN multiple-source multicast network simplifies to the set of rate tuples such that

$$\begin{aligned} \sum_{k \in \mathcal{S}} R_k \leq & \frac{1}{2} \log \left| I + \frac{P}{2} G(\mathcal{S}) G(\mathcal{S})^T \right| \\ & + \frac{1}{2} \min\{|\mathcal{S}|, |\mathcal{S}^c|\} \log(2|\mathcal{S}|) \end{aligned} \quad (7)$$

for all $\mathcal{S} \subseteq [1:N]$ with $\mathcal{S}^c \cap \mathcal{D} \neq \emptyset$. To show this, first note that the cutset outer bound (2) continues to hold with the set of input distributions satisfying $\mathsf{E}(X_k^2) \leq P$, $k \in [1:N]$. For each $\mathcal{S} \subseteq [1:N]$ such that $\mathcal{S}^c \cap \mathcal{D} \neq \emptyset$, we can further loosen the cutset outer bound as

$$\begin{aligned}
R(\mathcal{S}) &\leq I(X(\mathcal{S}); Y(\mathcal{S}^c)|X(\mathcal{S}^c)) \\
&= h(Y(\mathcal{S}^c)|X(\mathcal{S}^c)) - h(Y(\mathcal{S}^c)|X^N) \\
&= h\big(G(\mathcal{S})X(\mathcal{S}) + Z(\mathcal{S}^c)|X(\mathcal{S}^c)\big) - h(Y(\mathcal{S}^c)|X^N) \\
&= h\big(G(\mathcal{S})X(\mathcal{S}) + Z(\mathcal{S}^c)\big) - h(Y(\mathcal{S}^c)|X^N) \\
&= \frac{1}{2}\log(2\pi e)^{|\mathcal{S}^c|}\left|I + G(\mathcal{S})K_{X(\mathcal{S})}G(\mathcal{S})^T\right| \qquad (8) \\
&\quad - \frac{|\mathcal{S}^c|}{2}\log(2\pi e) \\
&\leq \frac{1}{2}\log\left|I + \mathrm{tr}(K_{X(\mathcal{S})})G(\mathcal{S})G(\mathcal{S})^T\right| \qquad (9) \\
&\leq \frac{1}{2}\log\left|I + |\mathcal{S}|P\cdot G(\mathcal{S})G(\mathcal{S})^T\right| \qquad (10) \\
&\leq \frac{1}{2}\log\left|2|\mathcal{S}|\cdot I + 2|\mathcal{S}|\frac{P}{2}\cdot G(\mathcal{S})G(\mathcal{S})^T\right| \\
&\leq \frac{1}{2}\log\left|I + \frac{P}{2}G(\mathcal{S})G(\mathcal{S})^T\right| + \frac{|\mathcal{S}^c|}{2}\log(2|\mathcal{S}|),
\end{aligned}$$

where $K_{X(\mathcal{S})}$ is the covariance matrix of $X(\mathcal{S})$, (9) follows since $\mathrm{tr}(K)I - K$ is positive semidefinite for any covariance matrix K [16, Theorem 7.7.3], and (10) follows since $\mathrm{tr}(K_{X(\mathcal{S})}) \leq |\mathcal{S}|P$, from the power constraint. By rewriting (10) as

$$\frac{1}{2}\log\left|I+|\mathcal{S}|P\cdot G(\mathcal{S})G(\mathcal{S})^T\right| = \frac{1}{2}\log\left|I+|\mathcal{S}|P\cdot G(\mathcal{S})^T G(\mathcal{S})\right|$$

and following similar steps, we also have

$$\begin{aligned}
R(\mathcal{S}) &\leq \frac{1}{2}\log\left|I + \frac{P}{2}G(\mathcal{S})^T G(\mathcal{S})\right| + \frac{|\mathcal{S}|}{2}\log(2|\mathcal{S}|) \\
&= \frac{1}{2}\log\left|I + \frac{P}{2}G(\mathcal{S})G(\mathcal{S})^T\right| + \frac{|\mathcal{S}|}{2}\log(2|\mathcal{S}|).
\end{aligned}$$

Next, we evaluate Theorem 1 for the Gaussian network. Let $Q = \emptyset$ and X_k, $k \in [1:N]$, be i.i.d. Gaussian with zero mean and variance P. Let

$$\hat{Y}_k = Y_k + \hat{Z}_k, \quad k \in [1:N],$$

where $\hat{Z}_k$, $k \in [1:N]$, are i.i.d. Gaussian with zero mean and unit variance. Then for each $\mathcal{S} \subseteq [1:N]$ such that $\mathcal{S}^c \cap \mathcal{D} \neq \emptyset$ and $d \in \mathcal{D}$,

$$\begin{aligned}
&I(\hat{Y}(\mathcal{S}); Y(\mathcal{S})|X^N, \hat{Y}(\mathcal{S}^c), Y_d) \\
&\quad\leq I(\hat{Y}(\mathcal{S}); Y(\mathcal{S})|X^N) \\
&\quad= h(\hat{Y}(\mathcal{S})|X^N) - h(\hat{Y}(\mathcal{S})|Y(\mathcal{S}), X^N) \\
&\quad= \frac{|\mathcal{S}|}{2}\log(4\pi e) - \frac{|\mathcal{S}|}{2}\log(2\pi e) \\
&\quad= \frac{|\mathcal{S}|}{2},
\end{aligned}$$

where the first inequality is due to the Markovity $(\hat{Y}(\mathcal{S}^c), Y_d) \to (X^N, Y(\mathcal{S})) \to \hat{Y}(\mathcal{S})$. Furthermore,

$$\begin{aligned}
&I(X(\mathcal{S}); \hat{Y}(\mathcal{S}^c), Y_d|X(\mathcal{S}^c)) \\
&\quad\geq I(X(\mathcal{S}); \hat{Y}(\mathcal{S}^c)|X(\mathcal{S}^c)) \\
&\quad= h(\hat{Y}(\mathcal{S}^c)|X(\mathcal{S}^c)) - h(\hat{Y}(\mathcal{S}^c)|X^N) \\
&\quad= \frac{1}{2}\log(2\pi e)^{|\mathcal{S}^c|}\left|2I + G(\mathcal{S})PG(\mathcal{S})^T\right| - \frac{|\mathcal{S}^c|}{2}\log(4\pi e) \\
&\quad= \frac{1}{2}\log\left|I + \frac{P}{2}G(\mathcal{S})G(\mathcal{S})^T\right|.
\end{aligned}$$

Therefore, by Theorem 1, a rate tuple $(R_1, \ldots, R_N)$ is achievable if $R(\mathcal{S}) < \frac{1}{2}\log\left|I + \frac{P}{2}G(\mathcal{S})G(\mathcal{S})^T\right| - \frac{|\mathcal{S}|}{2}$ for all $\mathcal{S} \subseteq [1:N]$ such that $\mathcal{S}^c \cap \mathcal{D} \neq \emptyset$.

Comparing the above outer and inner bounds completes the proof of Theorem 4.

B. Two-way relay channel

Consider the AWGN two-way relay channel

$$\begin{aligned}
Y_1 &= g_{21}X_2 + g_{31}X_3 + Z_1, \\
Y_2 &= g_{12}X_1 + g_{32}X_3 + Z_2, \\
Y_3 &= g_{13}X_1 + g_{23}X_2 + Z_3,
\end{aligned}$$

in which source nodes 1 and 2 wish to exchange messages reliably with the help of relay node 3. Various coding schemes for this channel have been investigated in [12], [13].

Rankov and Wittneben [12] showed that the amplify–forward (AF) coding scheme results in the inner bound on the capacity region that consists of all rate pairs (R_1, R_2) such that

$$R_k < \frac{1}{2}\log\left(\frac{a_k + \sqrt{a_k^2 - b_k^2}}{2}\right), \quad k \in \{1, 2\}$$

for some $\alpha \leq \sqrt{P/(g_{13}^2 P + g_{23}^2 P + 1)}$, where $a_1 := 1 + \frac{P(g_{12}^2 + \alpha^2 g_{32}^2 g_{13}^2)}{g_{32}^2\alpha^2 + 1}$, $a_2 := 1 + \frac{P(g_{21}^2 + \alpha^2 g_{31}^2 g_{23}^2)}{g_{31}^2\alpha^2 + 1}$, $b_1 := \frac{2P\alpha g_{32}g_{13}g_{12}}{g_{32}^2\alpha^2 + 1}$, and $b_2 := \frac{2P\alpha g_{31}g_{23}g_{21}}{g_{31}^2\alpha^2 + 1}$. They also showed that an extension of the original compress–forward (CF) coding scheme for the relay channel to the two-way relay channel results in the following inner bound on the capacity region that consists of all rate pairs (R_1, R_2) such that

$$\begin{aligned}
R_1 &< \mathrm{C}\left(\frac{g_{13}^2 P + (1+\sigma^2)g_{12}^2 P}{1+\sigma^2}\right), \\
R_2 &< \mathrm{C}\left(\frac{g_{23}^2 P + (1+\sigma^2)g_{21}^2 P}{1+\sigma^2}\right)
\end{aligned}$$

for some

$$\begin{aligned}
\sigma^2 &\geq \frac{(1+g_{12}^2 P)(1+g_{13}^2 P) - (g_{12}g_{13}P)^2}{\min\{g_{32}^2, g_{31}^2\}P}, \\
\sigma^2 &\geq \frac{(1+g_{21}^2 P)(1+g_{23}^2 P) - (g_{21}g_{23}P)^2}{\min\{g_{32}^2, g_{31}^2\}P}.
\end{aligned}$$

Specializing Theorem 2 to the AWGN two-way relay channel by setting $Q = \emptyset$ and $\hat{Y}_3 = Y_3 + \hat{Z}$ with $\hat{Z} \sim \mathrm{N}(0, \sigma^2)$, this inner bound simplifies to the set of rate pairs (R_1, R_2)

such that

$$R_1 < \mathrm{C}(g_{12}^2 P + g_{32}^2 P) - \mathrm{C}(1/\sigma^2),$$
$$R_1 < \mathrm{C}\left(\frac{g_{13}^2 P + (1+\sigma^2) g_{12}^2 P}{1+\sigma^2}\right),$$
$$R_2 < \mathrm{C}(g_{21}^2 P_1 + g_{31}^2 P) - \mathrm{C}(1/\sigma^2),$$
$$R_2 < \mathrm{C}\left(\frac{g_{23}^2 P + (1+\sigma^2) g_{21}^2 P}{1+\sigma^2}\right)$$

for some $\sigma^2 > 0$.

In Figure 1, we compare the performance of noisy network coding (Theorem 2) to AF and CF for the case $g_{12} = g_{21} = 1$, $g_{13} = g_{31} = d^{-\gamma/2}$, and $g_{23} = g_{32} = (1-d)^{-\gamma/2}$, where $d \in [0, 1/2]$ is the location of the relay node between nodes 1 and 2 (which are unit distance apart) and $\gamma = 3$. Note that noisy network coding outperforms the other two schemes, coinciding with the compress–forward only when the relay is midway between nodes 1 and 2 ($d = 1/2$) and when it is collocated with node 1 ($d = 0$).

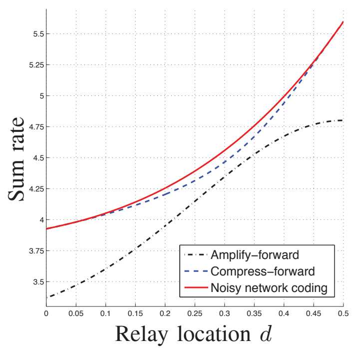

Fig. 1. Comparison of coding schemes for $P = 10$.

C. Interference relay channel

Consider the AWGN interference relay channel with orthogonal receiver components in Figure 2. The channel outputs are

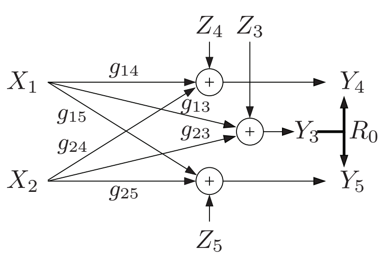

Fig. 2. AWGN interference relay channel.

$Y_j = g_{1j} X_1 + g_{2j} X_2 + Z_j, \quad j = 3, 4, 5$, where g_{ij} is the channel gain of link (i, j). Source node 1 wishes to send a message to destination node 4, while source node 2 wishes to send a message to destination node 5. Relay node 3 helps the communication of this interference channel by sending some information about Y_3 over a common noiseless link of rate R_0 to both destination nodes.

Djeumou, Belmaga, and Lasaulce [14], and Razaghi and Yu [15] showed that an extension of the original compress–forward coding scheme for the relay channel to the interference relay channel results in the inner bound on the capacity region that consists of all rate pairs (R_1, R_2) such that

$$R_1 < \mathrm{C}\left(\frac{(g_{13}^2 + (1+\sigma^2) g_{14}^2) P + (g_{23} g_{14} - g_{24} g_{13})^2 P^2}{1 + \sigma^2 + (g_{23}^2 + (1+\sigma^2) g_{24}^2) P}\right),$$
$$R_2 < \mathrm{C}\left(\frac{(g_{23}^2 + (1+\sigma^2) g_{25}^2) P + (g_{13} g_{25} - g_{15} g_{23})^2 P^2}{1 + \sigma^2 + (g_{13}^2 + (1+\sigma^2) g_{15}^2) P}\right)$$

for some

$$\sigma^2 \geq \frac{1}{2^{2R_0} - 1} \cdot \max\left\{ \frac{(g_{13} g_{24} - g_{23} g_{14})^2 P^2 + a_1}{(g_{14}^2 P + g_{24}^2 P + 1)}, \frac{(g_{13} g_{25} - g_{23} g_{15})^2 P^2 + a_2}{(g_{15}^2 P + g_{25}^2 P + 1)} \right\},$$

where $a_1 := (g_{13}^2 + g_{14}^2) P + (g_{23}^2 + g_{24}^2) P + 1$ and $a_2 := (g_{13}^2 + g_{15}^2) P + (g_{23}^2 + g_{25}^2) P + 1$. Razaghi and Yu [15] generalized the hash–forward coding scheme [17], [18] for the relay channel to the interference relay channel, in which the relay sends the bin index (hash) of its noisy observation and destination nodes use list decoding. This generalized hash–forward scheme gives the inner bound on the capacity region that consists of the set of rate pairs (R_1, R_2) such that

$$R_1 < \mathrm{C}\left(\frac{g_{14}^2 P}{g_{24}^2 P + 1}\right) + R_0 - \mathrm{C}\left(\frac{(g_{23}^2 + g_{24}^2) P + 1}{(g_{24}^2 P + 1)\sigma^2}\right),$$
$$R_2 < \mathrm{C}\left(\frac{g_{25}^2 P}{g_{15}^2 P + 1}\right) + R_0 - \mathrm{C}\left(\frac{(g_{13}^2 + g_{15}^2) P + 1}{(g_{15}^2 P + 1)\sigma^2}\right)$$

for some $\sigma^2 > 0$ satisfying

$$\sigma^2 \leq \frac{1}{2^{2R_0} - 1} \cdot \min\left\{ \frac{(g_{13} g_{24} - g_{23} g_{14})^2 P^2 + a_1}{(g_{14}^2 P + g_{24}^2 P + 1)}, \frac{(g_{13} g_{25} - g_{23} g_{15})^2 P^2 + a_2}{(g_{15}^2 P + g_{25}^2 P + 1)} \right\},$$

where a_1 and a_2 are the same as above.

Specializing Theorem 2 by setting $\hat{Y}_3 = Y_3 + \hat{Z}$ with $\hat{Z} \sim \mathrm{N}(0, \sigma^2)$ gives the inner bound that consists of all rate pairs (R_1, R_2) such that

$$R_1 < \mathrm{C}(g_{14}^2 P) + R_0 - \mathrm{C}(1/\sigma^2),$$
$$R_1 < \mathrm{C}\left(\frac{(g_{13}^2 + (1+\sigma^2) g_{14}^2) P}{1+\sigma^2}\right),$$
$$R_2 < \mathrm{C}(g_{25}^2 P) + R_0 - \mathrm{C}(1/\sigma^2),$$
$$R_2 < \mathrm{C}\left(\frac{(g_{23}^2 + (1+\sigma^2) g_{25}^2) P}{1+\sigma^2}\right),$$
$$R_1 + R_2 < \mathrm{C}((g_{14}^2 + g_{24}^2) P) + R_0 - \mathrm{C}(1/\sigma^2),$$
$$R_1 + R_2 < \mathrm{C}\left(\frac{aP + (1+\sigma^2)(g_{14}^2 + g_{24}^2) P + b_1^2 P^2}{1+\sigma^2}\right),$$
$$R_1 + R_2 < \mathrm{C}((g_{15}^2 + g_{25}^2) P) + R_0 - \mathrm{C}(1/\sigma^2),$$
$$R_1 + R_2 < \mathrm{C}\left(\frac{aP + (1+\sigma^2)(g_{25}^2 + g_{15}^2) P + b_2^2 P^2}{1+\sigma^2}\right),$$

where $a := g_{13}^2 + g_{23}^2$, $b_1 := g_{13} g_{24} - g_{23} g_{14}$, and $b_2 :=$

$g_{23}g_{15} - g_{13}g_{25}$, for some $\sigma^2 > 0$. By the same choice of $\hat{Y}_3$, the inner bound in Theorem 3 can be specialized to the set of rate pairs (R_1, R_2) such that

$$R_1 < \mathrm{C}\left(\frac{g_{14}^2 P}{g_{24}^2 P + 1}\right) + R_0 - \mathrm{C}\left(\frac{(g_{23}^2 + g_{24}^2)P + 1}{(g_{24}^2 P + 1)\sigma^2}\right),$$
$$R_1 < \mathrm{C}\left(\frac{(g_{13}^2 + (1+\sigma^2)g_{14}^2)P + (g_{23}g_{14} - g_{24}g_{13})^2 P^2}{1 + \sigma^2 + (g_{23}^2 + (1+\sigma^2)g_{24}^2)P}\right),$$
$$R_2 < \mathrm{C}\left(\frac{g_{25}^2 P}{g_{15}^2 P + 1}\right) + R_0 - \mathrm{C}\left(\frac{(g_{13}^2 + g_{15}^2)P + 1}{(g_{15}^2 P + 1)\sigma^2}\right),$$
$$R_2 < \mathrm{C}\left(\frac{(g_{23}^2 + (1+\sigma^2)g_{25}^2)P + (g_{13}g_{25} - g_{15}g_{23})^2 P^2}{1 + \sigma^2 + (g_{13}^2 + (1+\sigma^2)g_{15}^2)P}\right)$$

for some $\sigma^2 > 0$.

In Figure 3, we compare noisy network coding (Theorems 2 and 3) to compress–forward (CF) and hash–forward (HF) in [15]. The curve representing noisy network coding depicts the maximum of achievable sum rates in Theorems 2 and 3. Note that, although not shown in the figure, Theorem 3 alone outperforms the other two schemes for all channel gains and power constraints. At high signal-to-noise ratio (SNR), Theorem 2 provides further improvement, since decoding other messages is a better strategy when interference is strong.

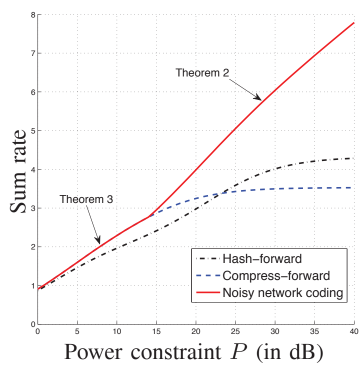

Fig. 3. Comparison of coding schemes for $g_{14} = g_{25} = 1$, $g_{15} = g_{24} = g_{13} = 0.5$, $g_{13} = 0.1$.

III. Concluding Remarks

We presented two extensions of noisy network coding and demonstrated that the new schemes can outperform previous network compress–forward schemes. The reasons are: first, the relays do not use Wyner–Ziv coding (no binning index to decode), second, the destinations are not required to decode the compression indices correctly, and third, simultaneous decoding over all blocks is used.

Another advantage of noisy network coding is that it performs generally well under high SNR conditions in the network. In addition, it is a robust and scalable scheme in the sense that the relay operations do not depend on the specific codebooks used by the sources and destinations or even the topology of the network. Noisy network coding, however, is not always the best strategy. For example, for a cascade of AWGN channels with low SNR, the optimal strategy is for the relay to decode the message and then forward it to the final destination. Noisy network coding can be further improved by combining with partial decode–forward [7] to obtain hybrid schemes similar to those in [7] and [19].

Acknowledgments

The work was supported in part by the DARPA ITMANET program, NSF CAREER grant CCF-0747111, and MKE/IITA IT R&D program 2008-F-004-02.

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